

Technical Report No. 178

SACLANT ASW
RESEARCH CENTRE

A GAME THEORY SOLUTION TO AN AIMING PROBLEM

by

HANS GEORG DIESS

15 OCTOBER 1970

NATO

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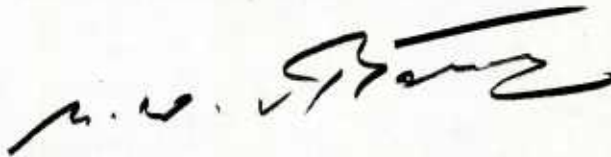
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A handwritten signature in black ink, appearing to read 'M.W. van Batenburg', is written over the printed name.

Ir M.W. van Batenburg
Director

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TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	1
INTRODUCTION	2
1. DEFINITION OF THE GAME	4
1.1 Strategies for the Attacker	4
1.2 Strategies for the Target	4
1.3 Pay-off	4
2. SOLUTIONS OF THE GAME	8
2.1 Introductory	8
2.2 Pure Attacker Strategy	9
2.3 Mixed Attacker Strategy	17
CONCLUSIONS	24
APPENDIX A	
THE OPTIMUM MIXED TARGET STRATEGY	25
APPENDIX B	
TRANSITION FROM PURE TO MIXED ATTACKER STRATEGIES	28

List of Figures

1. Probability area	3
2. Pay-off — general	5
3. Pay-off — simplified	6
4. Example for $u < 1$	8
5. Dominating strategy	10
6. Example of pay-off	11
7. Max-min	12
8. Min-max	14
9. Optimum strategy X_+	15
10. Types of strategies	17
11. Pay-off for strategies x_1 and x_2	19
A.1 Mixed optimum target strategy: iso-probability curves	27
B.1 Geometry for two-aimpoint strategy	29

A GAME THEORY SOLUTION TO AN AIMING PROBLEM

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ABSTRACT

This paper discusses the game theory solution to the following aiming problem:

An attacker receives information about the location of a target and launches a weapon. It is assumed that, at the moment it is supposed to be hit, the target may be anywhere within an annulus with radii R_1 , R_2 , which depend on the weapon delivery time and the target evasion manoeuvres. Using a polar coordinate system R, θ , the assumption is made that the target is uniformly distributed in θ , but chooses R between the limits R_1 and R_2 in order to maximize its chance of escape. The attacker will then distribute the weapon aimpoints uniformly in the angle θ (over its range $0, 2\pi$). For a single weapon $\theta = 0$ is chosen arbitrarily and the problem is reduced to the choice of the radial coordinate X for the aimpoint. The pay-off for this two-person game, where both players have continuous strategies, is expressed by the probability that the distance of the target from the impact point is less than the effective damage radius, e , of the weapon. Pure and mixed strategy solutions are discussed and conditions are derived for the normalized parameters $2e/(R_2 - R_1)$ and $R_2/(R_2 - R_1)$ that allow one to determine the type of strategies for a given set of values of the parameters e, R_1 and R_2 .

INTRODUCTION

In warfare, as well as in many games, a successful conclusion is often the result of out-guessing an opponent. The ability to out-guess the opponent usually increases with experience and a thorough knowledge of the moves available to both sides.

Although complex situations are often not susceptible to mathematical analysis, many of the simpler parts of complex situations are, and the results of such analyses can often be very useful in the determination of the best manoeuvres, given certain conditions.

This paper, dealing with a rather simple geometry of possible target location, weapon impact point and effective range, illustrates that for both target and attacker the best choice of manoeuvres is well defined.

The paper concerns the interplay between an attacker and a target that has been detected by the attacker at time zero. It is assumed that the attacker has no information about the target's motion at the instant of detection and that he is not sure that it will be possible to determine the location of the target at some later time. Therefore a weapon is launched with the minimum delay in order to reduce the uncertainty about the target's position at the instant of the attack. Given the time between detection and the instant of weapon impact, the target can be expected to be in an annulus centred at the point of detection. This is shown in Fig. 1, in which R_1 and R_2 are the distances the target can travel using its minimum and maximum evasion speeds, respectively. Using a polar-coordinate system, the position of the target can be described by a radius R and an angle θ , with the restrictions:

$$R_1 \leq R \leq R_2$$

The area defined by the above inequalities will be called the probability area.

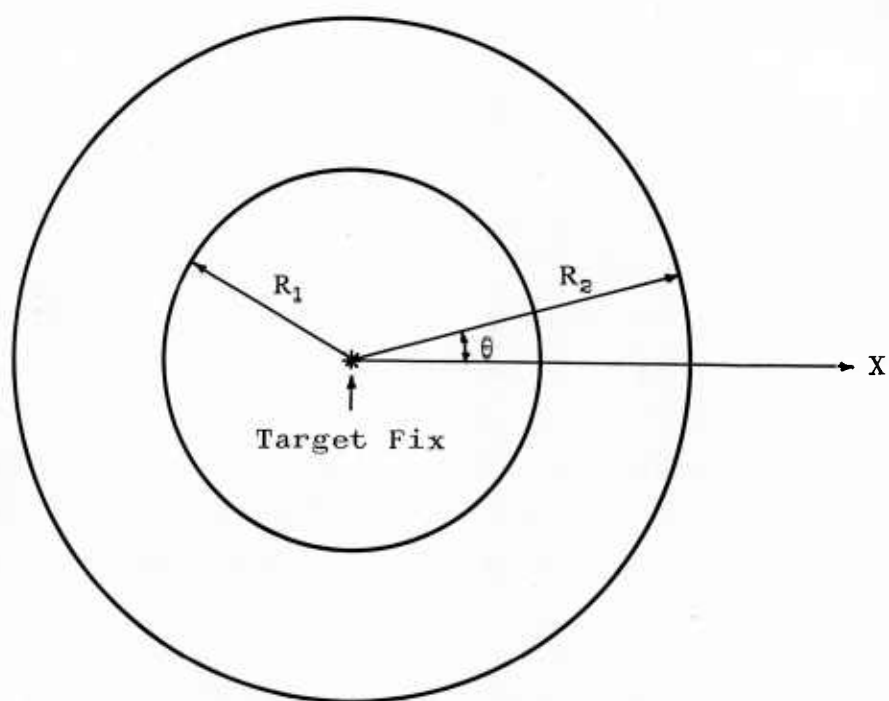


FIG. 1 PROBABILITY AREA

For the attacker it is assumed that one weapon is fired at an aimpoint having coordinates X, θ . Because of the symmetry of the probability area the coordinate θ is chosen arbitrarily as $\theta \approx 0$.

The following discussion will therefore be directed to the problem of the choice of the optimum X-coordinate of the aimpoint and the optimum evasion speed of the target.

1. DEFINITION OF THE GAME

To describe the game being studied it will be necessary to specify the strategies of the two players and the pay-off as a function of these strategies.

1.1 Strategies for the Attacker

As already pointed out in the introduction, the strategies of the attacker concern the choice of the value of X : the distance between the aimpoint and the point of detection of the target.

1.2 Strategies for the Target

The target has, in principle, a two-dimensional set of strategies, namely the choices of the polar coordinates θ and R of its position in the probability area at the instant of weapon impact.

It is assumed that θ is uniformly distributed in the interval

$$0 \leq \theta \leq 2\pi .$$

However, the distance R of the target from the point of its detection is chosen from the interval

$$R_1 \leq R \leq R_2 .$$

1.3 Pay-off

Consider now a target at a distance R from the point of detection and at a distance D from the aimpoint. Knowing the distribution of the impact points of the weapon around the aimpoint and the effective damage radius, e , of the weapon, the probability that the distance between the target and the impact

point of the weapon is less than e can be calculated. This probability will be used as the pay-off for the pair R, X of strategies for target and attacker, respectively.

Figure 2 shows the pay-off P as a function of the offset distance D for various values of the standard deviation σ of the aiming error distribution which is assumed to be circular normal. It is clear that for $\sigma=0$ a step function is obtained. If the aiming error is small then the pay-off will decrease rapidly from one to zero in the vicinity of the value $D=e$. Finally, for a large error σ the pay-off decreases slowly with D .

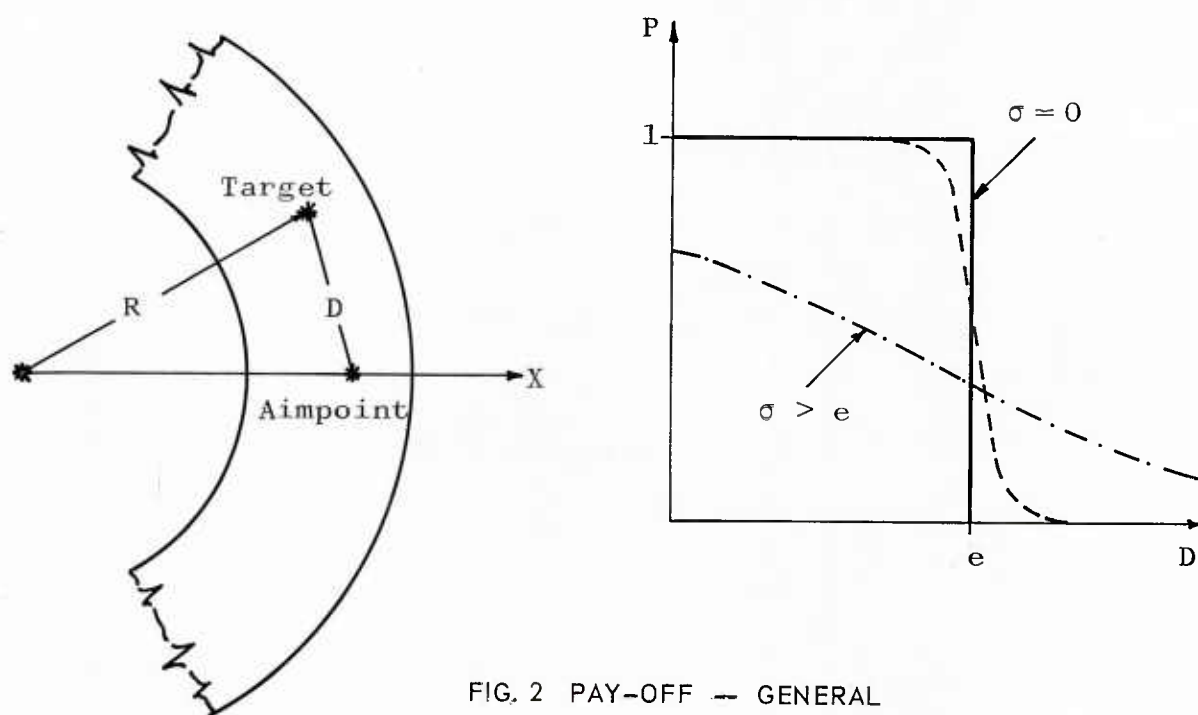


FIG. 2 PAY-OFF -- GENERAL

The pay-off can be approximated by a step function such as the one shown in Fig. 2. If then the aiming errors are small, e' will be very close to e and the assumption $\sigma=0$ is justified. For large aiming errors the impact points of the weapon will be distributed over a large area and changes in the aimpoint have little effect on the pay-off. For the following calculation of the optimum aimpoint the assumption will then be made that the standard deviations of the aiming errors are zero.

Figure 3 illustrates the simplified definition of the pay-off. Because the targets are distributed uniformly in angle, and since only targets within a distance e from the aimpoint are successfully attacked, the pay-off, or the proportion of the targets within the circle with radius e , can be expressed by a geometric probability. Suppose the target is at a distance R from the point of detection, then

$$P = \frac{2\theta R}{2\pi R} = \frac{\theta}{\pi} \quad [\text{Eq. 1}]$$

is the pay-off for a given aimpoint. We shall use θ alone as the pay-off as $1/\pi$ is only a proportional factor.

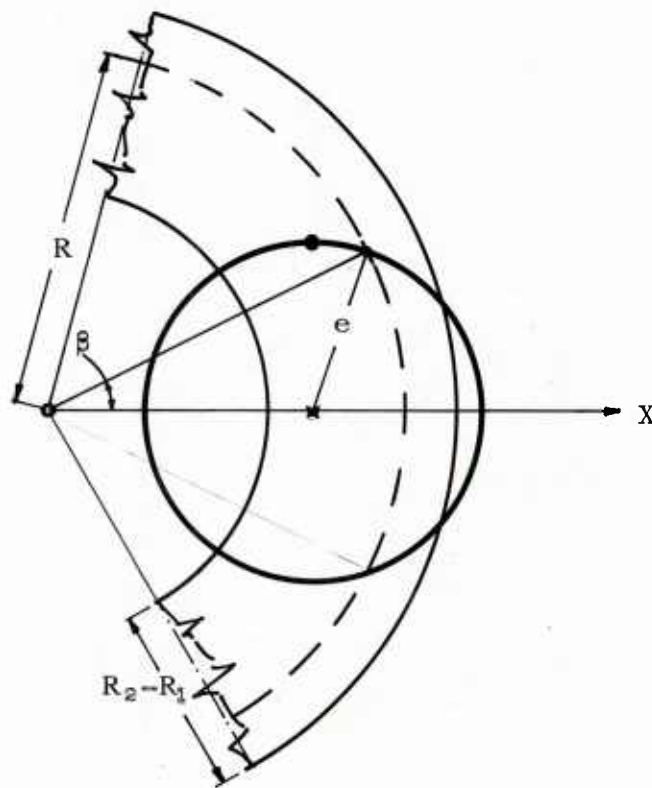


FIG. 3 PAY-OFF — SIMPLIFIED

The game for which a solution will be determined can then be described as follows:

The two players, the target and attacker, have continuous strategies, namely the selections of

R = distance travelled by the target, $R_1 \leq R \leq R_2$ and
 X = aimpoint coordinate for the attacker's weapon, $X \geq 0$.

The pay-off for a given pair of strategies is defined by Eq. 1. Since the pay-off is proportional to the angle β with

$$\beta(R, X) = \begin{cases} \arccos \left(\frac{R^2 + X^2 - e^2}{2RX} \right) & \text{if } |R - X| < e \\ 0 & \text{if } |R - X| > e \end{cases} \quad [\text{Eq. 2}]$$

strategies are chosen such that β is maximized by the attacker and minimized by the target.

2. SOLUTIONS OF THE GAME

2.1 Introductory

The aim of this chapter is to solve the game for pure and mixed attacker strategies and to derive conditions for the transition from one case to the other in terms of the parameters e , R_1 and R_2 . The dimensionality of the problem can be reduced by using the dimensionless variables

$$u = \frac{2e}{R_2 - R_1} \quad v = \frac{R_2}{R_2 - R_1} \quad [\text{Eq. 3}]$$

The advantage of using this particular set of variables soon becomes obvious. It should be noticed that always $v > 1$.

From Fig. 4 it is clear that with $u < 1$ for any pure strategy X there will be an interval of values for the distance R that the target can choose such that the pay-off to the attacker is zero. Therefore $u < 1$ is a sufficient condition for mixed attacker strategies. It will be shown, however, that the transition between pure and mixed attacker strategies will occur at values of u somewhat greater than unity.

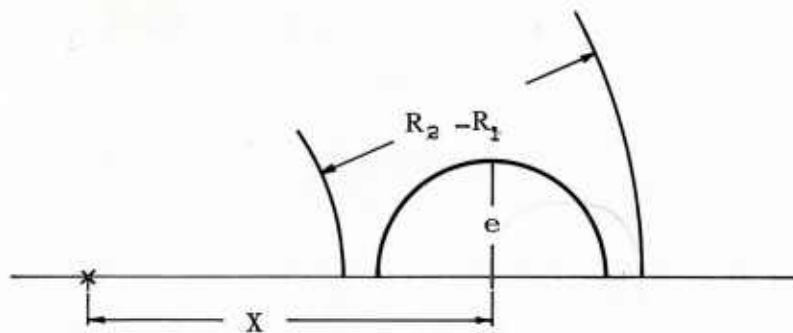


FIG. 4 EXAMPLE FOR $u < 1$

Also the assumption is made that $e < R_2$, because otherwise complete coverage of the probability area is obtained by the pure strategy $X = 0$.

2.2 Pure Attacker Strategy

The game is solved first for pure attacker strategies, i.e., it is assumed that the attacker uses only one aimpoint. As discussed above this requires that $u > 1$.

For the following discussion it is useful to define an upper limit X_2 for the attacker strategy X in the sense that X_2 dominates all strategies $X > X_2$. In this game dominance means that $\beta(X_2) > \beta(X)$ for all values R of the target strategy.

From Eq. 2 one finds that the derivative of β with respect to X is proportional to $X^2 - R^2 + e^2$ and hence the maximum of β for given R is reached for $X_R = \sqrt{R^2 - e^2}$.

Let $X_2 = \sqrt{R_2^2 - e^2}$ and note that $X_R \leq X_2$. Then it follows from the convexity of $\beta(X)$ that $\beta(X_R) \geq \beta(X_2) \geq \beta(X)$ for $X \geq X_2$. This is true for all $R \leq R_2$, i.e., X_2 dominates all strategies $X > X_2$.

The strategies actually used by the players then can be restricted to the intervals

$$0 \leq X \leq X_2 \quad \text{for the attacker}$$

and

$$R_1 \leq R \leq R_2 \quad \text{for the target}$$

where

$$X_2 = \sqrt{R_2^2 - e^2} . \quad [\text{Eq. 4}]$$

Figure 5 illustrates this special strategy X_2 .

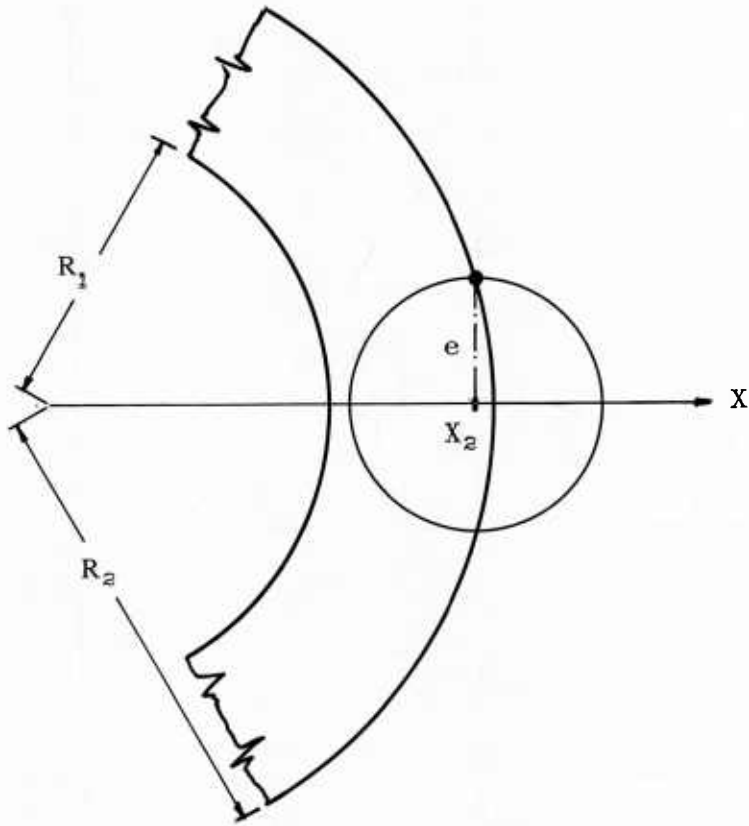


FIG. 5 DOMINATING STRATEGY

Since in this section the game will be solved for pure attacker strategies X , the necessary assumption is that

$$u > 1$$

because otherwise the pay-off would be zero for a target behaving in an optimum way, as pointed out previously. Furthermore, one notices when using Eq. 2 that β is different from zero for all X and R satisfying the condition

$$|R - X| < e \quad . \quad [\text{Eq. 5}]$$

As the inequality $u > 1$ is equivalent to

$$R_2 - e < R_1 + e \quad [\text{Eq. 6}]$$

[see Eq. 3], it follows from Eqs. 5 and 6 that β is different from zero for all values of R if X is in the interval

$$R_2 - e < X < R_1 + e \quad . \quad [\text{Eq. 7}]$$

2.2.1 Max - Min

The angle β as a function of the target strategy R is shown in Fig. 6 for several aimpoint coordinates X with $X \leq X_2$ where X_2 is defined in Eq. 4.

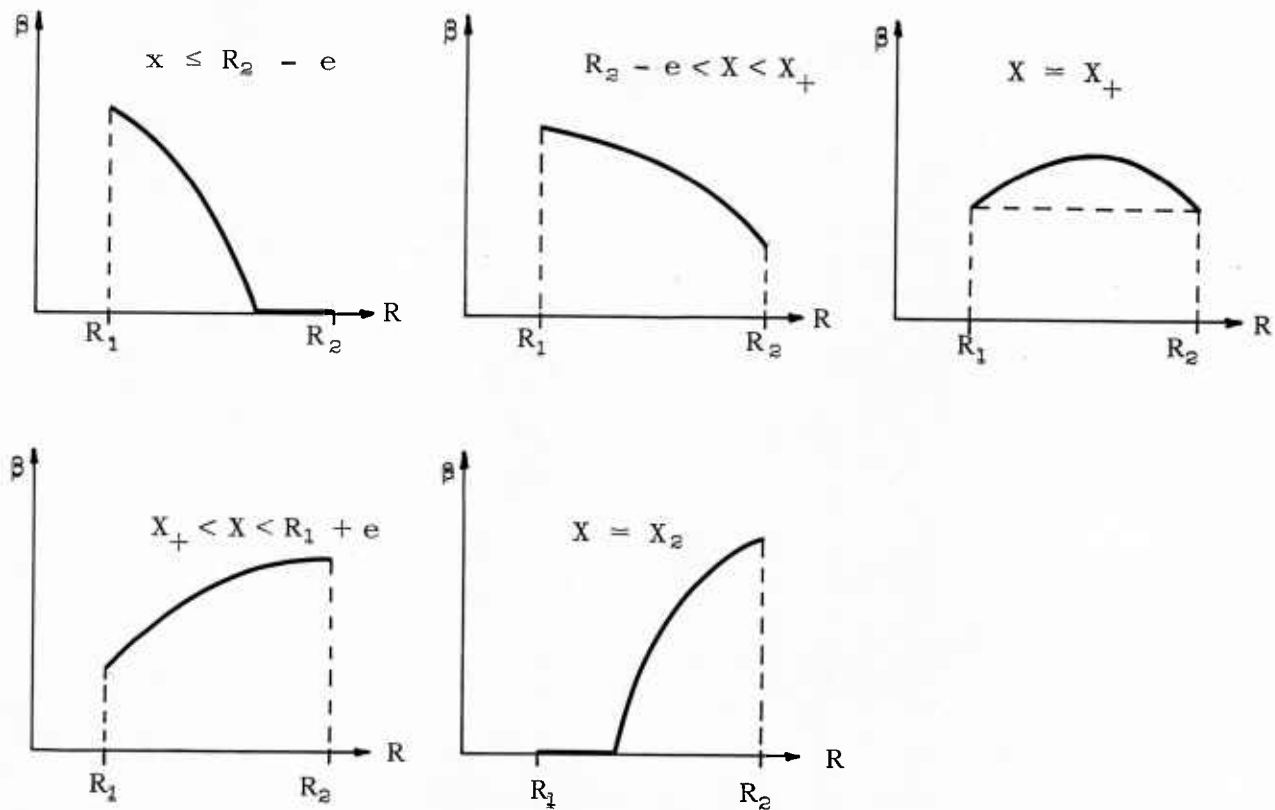


FIG. 6 EXAMPLE OF PAY-OFF

The characteristic of the strategy X_+ is that the pay-offs for both R_1 and R_2 are equal:

$$\beta(R_1, X_+) = \beta(R_2, X_+) \quad .$$

Using the notations

$$\beta_i(X) = \beta(R_i, X), \quad i = 1, 2$$

the following statement can be made.

If the target minimizes the pay-off β for a given aimpoint X , it can be seen from Fig. 6 that $\text{MIN } \beta$ is determined by the following expression:

$$\text{MIN } \beta = \begin{cases} 0, & X \leq R_2 - e \\ \beta_2(X), & R_2 - e < X \leq X_+ \\ \beta_1(X), & X_+ \leq X < \min(R_1 + e, X_2) \\ 0, & \min(R_1 + e, X_2) \leq X \leq X_2 \end{cases} \quad [\text{Eq. 8}]$$

A plot of $\text{MIN } \beta$ as a function of the attacker strategy X is shown in Fig. 7. The dashed curves indicate $\beta_1(X)$ for $X < X_+$ and $\beta_2(X)$ for $X > X_+$. Figure 7 clearly shows that $\text{MIN } \beta$ has a maximum for that attacker strategy X where

$$\beta_1(X) = \beta_2(X) \quad ,$$

which is the case for $X = X_+$ and therefore

$$\text{MAX-MIN } \beta = \beta(X_+) = \beta_+ \quad . \quad [\text{Eq. 9}]$$

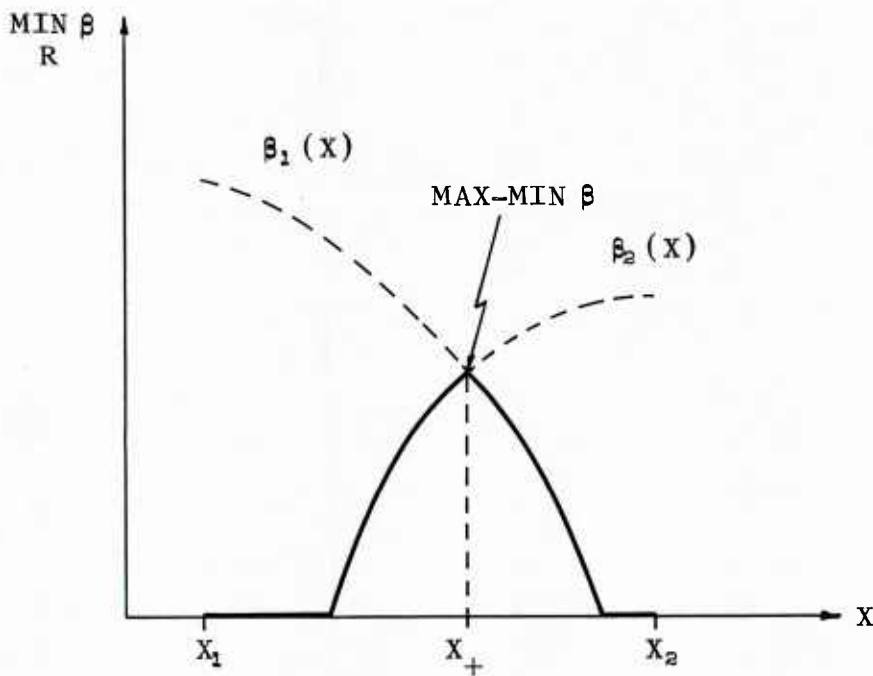


FIG. 7 MAX - MIN

2.2.2 Min - Max

The game is solved if it can be shown that $\text{MIN-MAX } \beta = \text{MAX-MIN } \beta = \beta(X_+)$. Attention is therefore concentrated on the interval $R_2 - e, R_1 + e$, which contains the strategy X_+ .

As pointed out before, $\beta(R, X)$ is different from zero for all values of X satisfying Eq. 7 and it is convex with respect to R as one can see from Fig. 6 and Eq. 2.

Suppose now the target uses a mixed strategy and let p_1 and p_2 be the probabilities of playing the strategies r_1 and r_2 , respectively. It follows from the convexity of β that

$$p_1 \beta(r_1, X) + p_2 \beta(r_2, X) \leq \beta(r, X)$$

for pure strategies r in the interval r_1, r_2 .

This is true for all pairs of strategies r_1, r_2 , hence also for R_1 and R_2 . This means that a mixture between the strategies R_1 and R_2 is better for the target than any pure strategy R , since the target is the minimizing player. Calling p the probability of R_1 and $1-p$ the probability of R_2 , the pay-off can be written as an expected value

$$E(\beta) = p\beta_1(X) + (1-p)\beta_2(X), 0 \leq p \leq 1, R_2 - e \leq X \leq R_1 + e. \quad [\text{Eq. 10}]$$

$E(\beta)$ is therefore a linear function of p . Figure 8 shows this relationship for values of X in the interval $R_2 - e \leq X \leq R_1 + e$.

By maximizing the expected pay-off $E(\beta)$ for any mixed strategy p of the target, $\text{MAX } E(\beta)$ becomes identical to the envelope of all straight lines that can be drawn in Fig. 8. Consequently, the derivative of $\text{MAX } E(\beta)$ with respect to p can be written as

$$\frac{d \text{MAX } E(\beta)}{d p} = \beta_1(X_p) - \beta_2(X_p), \quad [\text{Eq. 11}]$$

where X_p maximizes $E(\beta)$ for a given p . Obviously, $\text{MAX } E(\beta)$

is minimized by the target if it uses a mixed strategy p that makes the right side of Eq. 11 equal to zero. This is the case for $X_p = X_+$ and therefore

$$\min_p \max_X E(\beta) = \beta(X_+) = \beta_+ \quad . \qquad \qquad \qquad [Eq. 12]$$

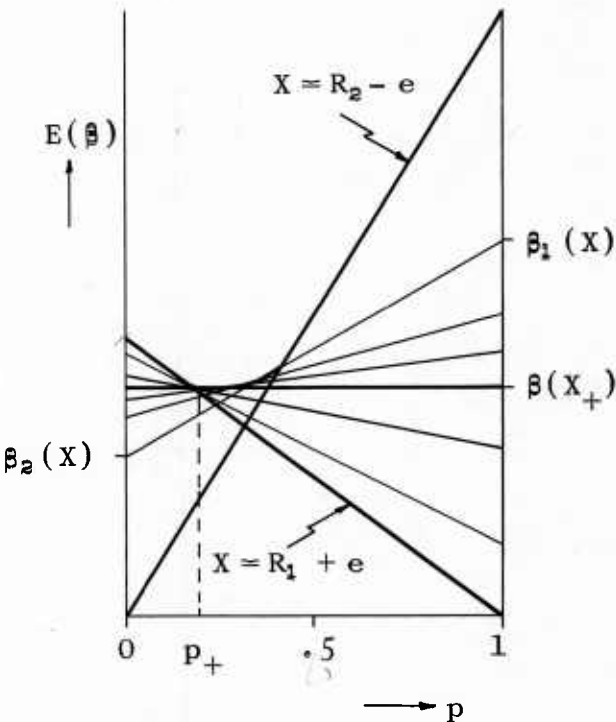


FIG. 8 MIN - MAX

2.2.3 Optimum Strategies

A comparison of Eqs. 9 and 12 yields

$$\max_X \min_p \beta = \min_p \max_X \beta = \beta_+ \quad ,$$

hence $\beta_+ = \beta(X_+)$ is the value of the game and X_+ is the optimum attacker strategy.

A geometrical interpretation of X_+ can be made using Fig. 9. The circle of radius e centred at X_+ intersects the two limits of the probability area at the same angle β_+ . This figure yields the relationship

$$X_+^2 = \left(\frac{R_1 + R_2}{2}\right)^2 + e^2 \left(\frac{R_2 - R_1}{2}\right)^2 ,$$

which has the solution

$$X_+ = \sqrt{R_1 R_2 + e^2} . \qquad \qquad \qquad [\text{Eq. 13}]$$

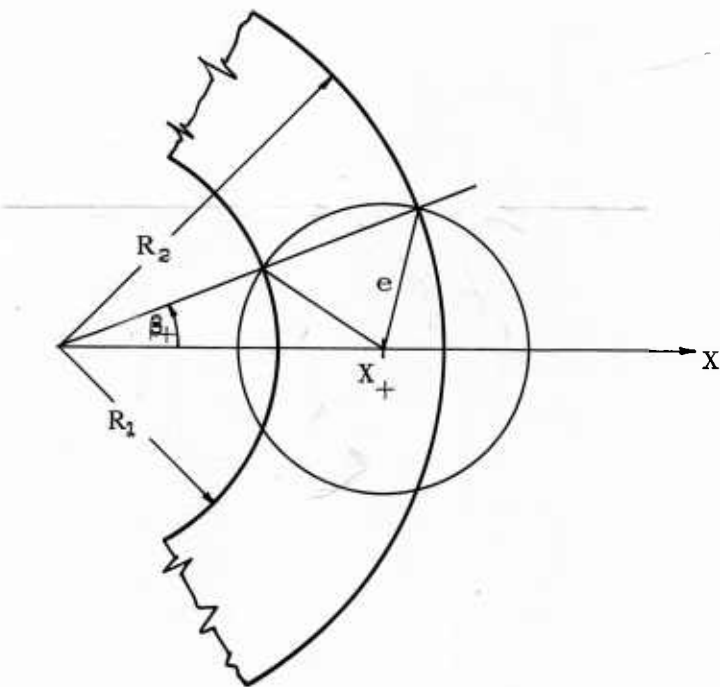


FIG. 9 OPTIMUM STRATEGY X_+

The mixed optimum strategy p_+ of the target is computed in Appendix A as a function of the dimensionless variables u and v :

$$\left(\frac{u}{2}\right)^2 = \frac{0.5 - p_+}{v - 1 + p_+} (v - 1) v . \qquad \qquad \qquad [\text{Eq. 14}]$$

As v is always greater than 1, the optimum mixed target strategy will be in the interval

$$0 \leq p_+ \leq 0.5$$

because otherwise $u^2 < 0$. This means that the target will use strategy R_2 more frequently than strategy R_1 .

2.2.4 Saddle-point Solution

For a saddle-point solution both the target and the attacker would have to have pure optimum strategies. This will be the case if $p_+ = 0$.

Equation 14 then yields

$$u^2 = 2v.$$

Replacing u and v by their definitions gives

$$\sqrt{R_1 R_2 + e^2} = \sqrt{R_2^2 - e^2}$$

The left-hand side of this equation is equal to X_+ [see Eq. 13] and the expression on the right is equal to X_2 [see Eq. 4].

Therefore $X_+ = X_2$, and the pure strategy (saddle-point) solution can be stated as follows:

The attacker's optimum pure strategy is to make $X = X_2$, the target's optimum pure strategy is to make $R = R_2$, and the value of the game is

$$\beta(R_2, X_2) = \arcsin (e/R_2).$$

The inequality $u^2 > 2v$ is equivalent to $X_+ > X_2$. However, it has been shown that X_2 dominates all strategies $X > X_2$. Therefore X_2 is the optimum strategy for all values of u and v with

$$u^2 \geq 2v.$$

The u - v domain is therefore divided into two regions: for $u^2 \geq 2v$ both target and attacker have pure optimum strategies.

This is shown in Fig. 10. The boundary of the area where both players have mixed optimum strategies will be discussed in the following section.

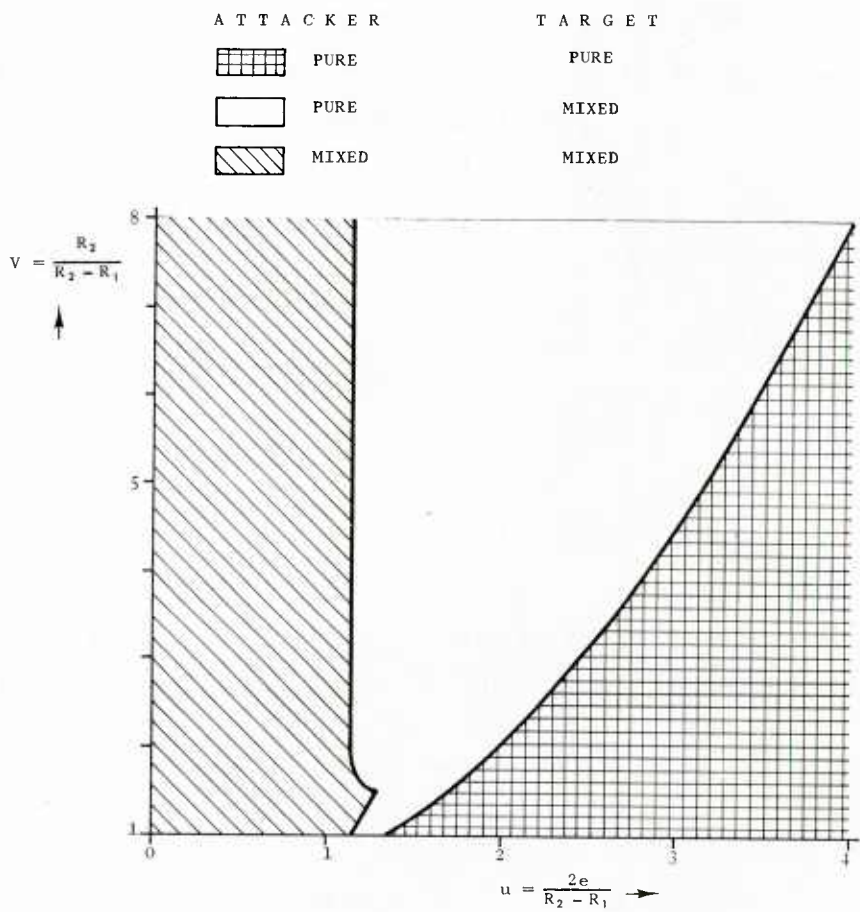


FIG. 10 TYPES OF STRATEGIES

2.3 Mixed Attacker Strategy

It is assumed in this section that the attacker is choosing the aimpoint from a set of two values x_1, x_2 , q being the probability of choosing x_1 . (Upper case letters refer to specific values of the variables.)

Again an assumption has to be made about the value of the variable

$$u = \frac{2e}{R_2 - R_1} \quad .$$

This time u has to be greater than 0.5, because for smaller values of u the target can always pick a strategy r such that the probability of being within weapon range is zero, irrespective of whether the attacker chooses x_1 or x_2 .

Because, in pure attacker strategies, the target had two strategies to choose from, it is reasonable to assume that the target now uses three strategies, r_1, r_2 and r_3 , with probabilities p_1, p_2 and p_3 .

Using the notation

$$\beta_{ij} = \beta(r_i, x_j) \quad i = 1, 2, 3, \quad j = 1, 2,$$

the expected pay-off for this game, where the attacker has two strategies x_1, x_2 and the target has three strategies r_1, r_2 and r_3 , can be written as

$$E = q \sum_{i=1}^3 p_i \beta_{i1} + (1-q) \sum_{i=1}^3 p_i \beta_{i2} \quad . \quad [\text{Eq. 15}]$$

The conditions under which this equation is valid are

$$q < 1 \quad \text{and} \quad \sum_{i=1}^3 p_i = 1 \quad .$$

For the target strategies r_i the inequality

$$R_1 \leq r_i \leq R_2$$

must hold where R_1 and R_2 define the boundary of the probability area.

Finally, it can be assumed without losing generality that

$$0 \leq x_1 < x_2 \quad .$$

For later reference the pay-off for this game as defined in Eq.15 is written in two different ways

$$E = \sum_{i=1}^3 p_i \beta_{i2} + q \sum_{i=1}^3 p_i (\beta_{i1} - \beta_{i2}) \quad [\text{Eq. 16}]$$

$$\sum_{i=1}^3 p_i [q \beta_{i1} + (1 - q) \beta_{i2}] \quad [\text{Eq. 17}]$$

The problem can then be stated as follows:

$$\text{Compute } M = \underset{q, x_1, x_2}{\text{MAX}} \underset{r_i, p_i}{\text{MIN}} E$$

and

$$m = \underset{r_i, p_i}{\text{MIN}} \underset{q, x_1, x_2}{\text{MAX}} E$$

under the conditions

$$q < 1 \quad \text{and} \quad \sum_{i=1}^3 p_i = 1$$

and find those values of q, x_1, x_2 and p_i, r_i for which

$$m = M.$$

The author did not succeed in solving the problem in this general form and therefore concentrated on the problem of transition from mixed to pure optimum attacker strategies. For this purpose some assumptions were made, which can be explained by reference to Fig. 11.

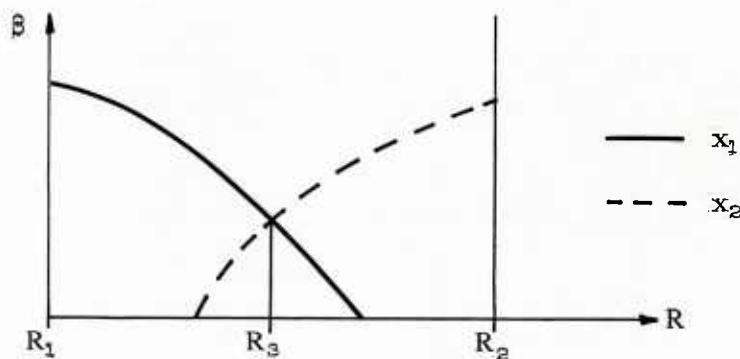


FIG. 11 PAY-OFF FOR STRATEGIES x_1 AND x_2

The figure shows the pay-off β for two aimpoints x_1, x_2 as a function of the target strategy r .

It is now assumed that the target uses the three particular strategies R_1, R_2 and R_3 , where R_1 and R_2 define the boundaries of the probability area and R_3 is determined by the equation

$$\beta(R, x_1) = \beta(R, x_2) \quad R \equiv R_3 \quad .$$

This strategy R_3 is likely to be used by the target as a pure strategy, if the pay-off is expected to be less than for any other strategy.

Under this assumption the value of the game is calculated as shown below.

2.3.1 Max - Min

Suppose the mixed attacker strategy, determined by x_1, x_2 and q , is given. Then with Eq. 17 the expected pay-off can be written as

$$E = \sum_{i=1}^3 p_i C_i$$

$$C_i = q \beta_{i1} + (1 - q) \beta_{i2} \quad . \quad [\text{Eq. 18}]$$

Since x_1, x_2 and q are given, the values of all C_i are known and positive.

One observes in Eq. 18 that E is a weighted mean of positive numbers and hence

$$E \geq \min_i C_i \quad .$$

By minimizing E , the target will choose the pure strategy $p_k = 1$, if C_k is the smallest of the three.

As we are interested in the three-way mixed strategy solution, it follows that all C_i must be equal, otherwise the pure strategy, which minimizes E , would be chosen. Therefore

$$\text{MIN } E = C_1 .$$

The attacker then can maximize this by a suitable choice of its aimpoint strategy, i.e. x_1 , x_2 and q , subject to the condition

$$C_1 = C_2 = C_3 .$$

To be more explicit, this condition is expressed in terms of the β_{ij} using the fact that $\beta_{31} = \beta_{32}$.

$$C_1 = C_2 \text{ yields } q = \frac{\beta_{22} - \beta_{12}}{\beta_{22} - \beta_{12} + (\beta_{11} - \beta_{21})} . \quad [\text{Eq. 19}]$$

This equation, together with $C_1 = C_3$, then determines a relationship between the β_{ij} :

$$\beta_{31} = \frac{\beta_{11} \beta_{22} - \beta_{12} \beta_{21}}{(\beta_{11} + \beta_{22}) - (\beta_{12} + \beta_{21})} . \quad [\text{Eq. 20}]$$

One can also verify that $C_1 = \beta_{31}$.

Calling $\beta^* = \text{MAX}_{x_1, x_2} \beta_{31}$ then

$$\text{MAX-MIN } E = \beta^* . \quad [\text{Eq. 21}]$$

As each β_{ij} in Eq. 20 is a function of x_1 and x_2 , maximizing β_{31} under the condition of Eq. 20 determines uniquely x_1 , x_2 and β^* .

The mixed strategy q can be calculated from Eq. 19.

2.3.2 Min - Max

Given a mixed strategy of the target with $p_i \neq 0$, $i = 1, 2, 3$, it follows from Eq. 16 that a solution with mixed strategies for the attacker can be obtained only if

$$S = \sum_{i=1}^3 p_i (\beta_{i1} - \beta_{i2}) = 0 \quad . \quad [\text{Eq. 22}]$$

If this were not true then the attacker, who tries to maximize E , would choose $q=1$ if $S > 0$ and $q=0$ if $S < 0$.

With Eqs. 16 and 22

$$\text{MAX } E = \sum_{i=1}^3 p_i \beta_{i2} \quad .$$

The target has to minimize this under the condition

$$S = 0 \quad \text{and} \quad \sum_{i=1}^3 p_i = 1 \quad .$$

The condition $S=0$ can be transformed into a relationship between p_1 and p_2 , the term with p_3 being cancelled because $\beta_{31} = \beta_{32}$. The result is

$$p_1 (\beta_{11} - \beta_{12}) = p_2 (\beta_{22} - \beta_{21}) \quad . \quad [\text{Eq. 23}]$$

In the expression for $\text{MAX } E$, p_3 is replaced by $1 - p_1 - p_2$ and then p_2 is substituted using Eq. 23. This leads to

$$\text{MAX } E = \beta_{31} + p_1 \left[\beta_{12} - \beta_{31} + (\beta_{22} - \beta_{31}) \frac{\beta_{11} - \beta_{12}}{\beta_{22} - \beta_{21}} \right] \quad .$$

The target can minimize this by choosing $p_1 = 1$ or $p_1 = 0$, depending on whether the expression in the bracket is negative or positive.

Again the argument is used that we seek a solution using mixed strategies and consequently the factor multiplying p_1 must be equal to zero. It turns out that this is true if Eq. 20 is fulfilled. Therefore,

$$\text{MIN} - \text{MAX } E = \beta_{31} \quad . \quad [\text{Eq. 24}]$$

2.3.3 Value of the Game

If the attacker maximizes β_{31} by an appropriate choice of x_1 and x_2 that simultaneously satisfy Eq. 20, one finds from Eqs. 21 and 24 that the value of the game is

$$\beta^* = \text{MAX}_{x_1, x_2} \beta_{31}$$

The question is when does β^* become equal to β_+ , i.e., when does the solution of the game with mixed attacker strategy change into one with pure attacker strategy? Appendix B briefly outlines the procedure for calculating the strategies x_1 and x_2 that maximize β_{31} and at the same time satisfy Eq. 20 and $\beta^* = \beta_+$. From a numerical calculation a relationship [see Fig. 10] between the variables u and v has been obtained that divides the $u-v$ domain into two regions: one with pure and one with mixed optimum strategies for the attacker.

It should be mentioned that there may be a set of target strategies for which the value of the game is smaller than calculated. Since the value of the game in the case of the pure attacker strategies, β_+ , approaches zero as u tends to 1, it follows that the boundary between pure and mixed optimum attacker strategies may be closer to $u=1$ than calculated.

CONCLUSION

The main result of this analysis is given by Fig. 10, from which a quick decision can be made about which of the possible strategies is best for any given values of the effective damage radius, e , of the weapon and of the distances, R_1 and R_2 , that the target can travel from its detection position when using minimum and maximum evasion speeds respectively.

APPENDICES

APPENDIX A

THE OPTIMUM MIXED TARGET STRATEGY

In Section 2.2 of the main text it was shown that in the vicinity of the solution of the game the expected pay-off to the attacker can be written as

$$E(\beta) = p \beta_1(X) + (1-p) \beta_2(X) \quad [\text{Eq. A.1}]$$

with $\beta_i(X) = \beta(R_i, X)$, $i = 1, 2$

$$\text{and } \beta(R, X) = \arccos \left(\frac{R^2 + X^2 - e^2}{2RX} \right) . \quad [\text{Eq. A.2}]$$

The value of the game was found to be

$$\beta_+ = \beta_1(X_+) = \beta_2(X_+) , \quad [\text{Eq. A.3}]$$

$$\text{where } X_+ = \sqrt{R_1 R_2 + e^2} . \quad [\text{Eq. A.4}]$$

The problem to be solved here is the calculation of the mixed optimum strategy p_+ of the target.

The derivative of the expected pay-off $E(\beta)$ with respect to X must vanish at the optimum, i.e.

$$\frac{dE}{dX} (p = p_+, X = X_+) = 0 .$$

Differentiating Eq. A.1 yields

$$p_+ = \frac{\left(\frac{d\beta_2}{dX} \right)_{X_+}}{\left(\frac{d\beta_2}{dX} \right)_{X_+} - \left(\frac{d\beta_1}{dX} \right)_{X_+}} . \quad [\text{Eq. A.5}]$$

The derivative of β_i ($i=1,2$) at X_+ can be calculated from Eq. A.2. One finds that the derivative of β_i at X_+ is proportional to

$$\frac{1}{R_i} - \frac{R_1 + R_2}{2 X_+^2}$$

and therefore

$$\frac{p_+}{1 - p_+} \approx - \frac{\frac{1}{R_2} - \frac{R_1 + R_2}{2 X_+^2}}{\frac{1}{R_1} - \frac{R_1 + R_2}{2 X_+^2}} \quad . \quad [\text{Eq. A.6}]$$

Multiplying the numerator and dominator of Eq. A.6 by $X_+^2 = R_1 R_2 + e^2$ and cancelling some terms yields

$$\frac{p_+}{1 - p_+} \approx \frac{0.5(R_2 - R_1) - e^2/R_2}{0.5(R_2 - R_1) + e^2/R_1} \quad . \quad [\text{Eq. A.7}]$$

At this point the dimensionless variables u and v are used. From their definition in Eq. 3 one finds

$$0.5(R_2 - R_1) \approx \frac{e}{u} , \quad \frac{e}{R_2} \approx \frac{u}{2v} , \quad \frac{e}{R_1} \approx \frac{u}{2(v-1)} \quad .$$

Replacing these terms in Eq. A.7 and multiplying through by u/e leads to

$$\frac{p_+}{1 - p_+} \approx \frac{1 - u^2/2v}{1 - u^2/2(v-1)}$$

This equation is finally solved for u , to give

$$\left(\frac{u}{2}\right)^2 \approx \frac{0.5 - p_+}{v - 1 + p_+} v \cdot (v - 1) \quad [\text{Eq. A.8}]$$

Plots of u versus v for given values of p_+ are shown in Fig. A.1.

Obviously, p_+ must lie in the interval

$$0 \leq p_+ \leq 0.5 \quad ,$$

otherwise $u^2 < 0$.

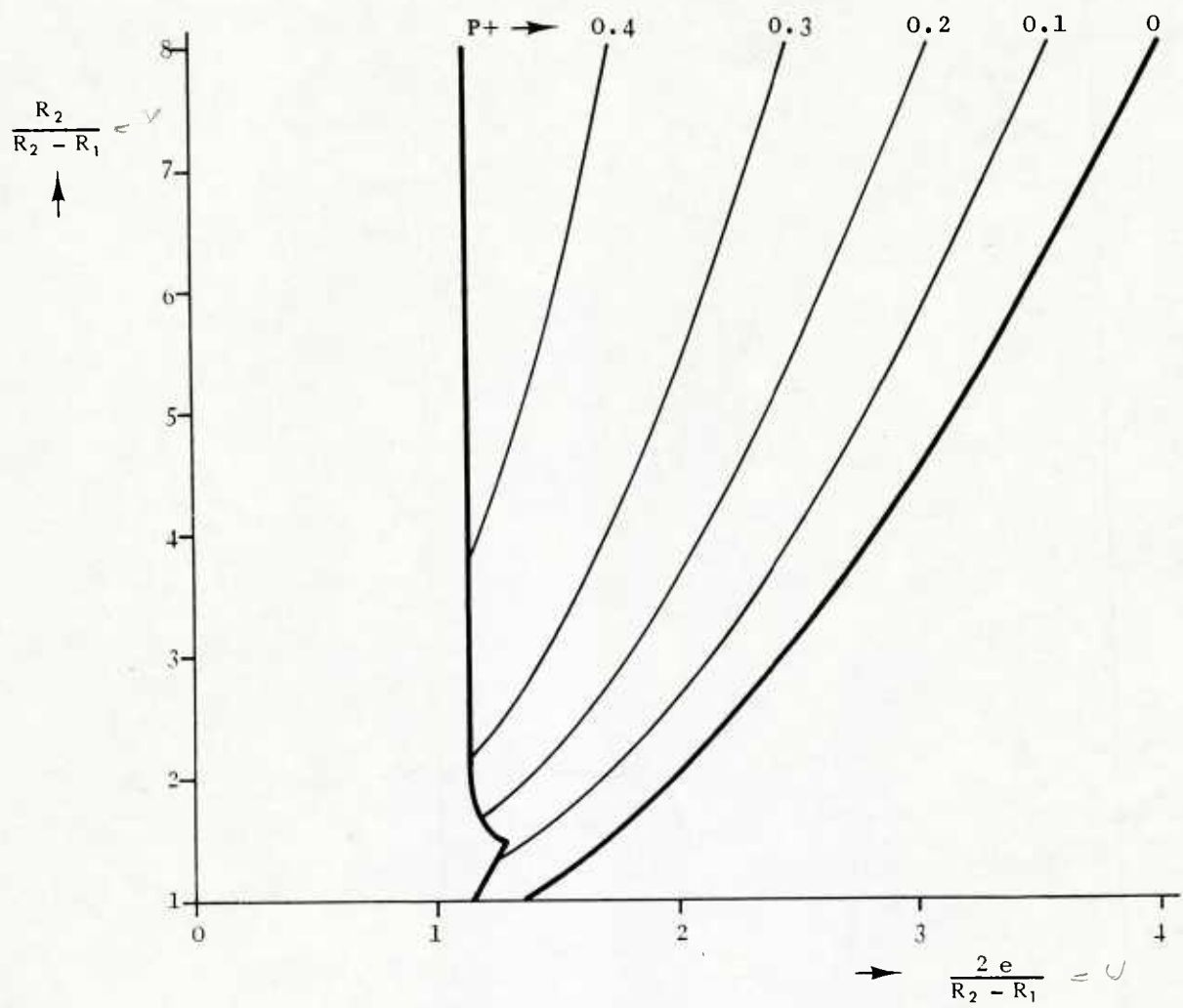


FIG. A.1 MIXED OPTIMUM TARGET STRATEGY: ISO-PROBABILITY CURVES

APPENDIX B

TRANSITION FROM PURE TO MIXED ATTACKER STRATEGIES

As pointed out in Section 2.3 of the main text, the game has a solution with a mixed optimum strategy for the attacker, if the equation

$$\beta_{31} = \frac{\beta_{11} \beta_{22} - \beta_{12} \beta_{21}}{(\beta_{11} + \beta_{22}) - (\beta_{12} + \beta_{21})} \quad [\text{Eq. B.1}]$$

is satisfied and β_{31} is maximized by the attacker.

Suppose two arbitrary aimpoints X_1, X_2 ($X_1 < X_2$) are given that determine the target strategy R and the angle β_{31} , as shown in Fig. B.1. The values of R and β_{31} can be computed as functions of the aimpoints X_1 and X_2 as follows.

Because the point with the coordinate R, β_{31} has the same distance e from the two aimpoints X_1 and X_2 , it is clear from Fig. B.1 that

$$\begin{aligned} R \cos \beta_{31} &= 0.5 (X_1 + X_2) \\ R \sin \beta_{31} &= \sqrt{e^2 - 0.25 (X_2 - X_1)^2} \end{aligned}$$

From those two equations the relationship

$$\cos \beta_{31} = \frac{0.5 (X_1 + X_2)}{\sqrt{X_1 X_2 + e^2}} \quad [\text{Eq. B.2}]$$

is obtained.

Furthermore, the angles β_{ij} can also be expressed as functions of the aimpoints by

$$\cos \beta_{ij} = \frac{R_i^2 + X_j^2 - e^2}{2 R_i X_j} ; \quad i, j = 1, 2 \quad [\text{Eq. B.3}]$$

and hence the right side of Eq. B.1 as a function of X_1 and X_2 .

It should be mentioned that in Eqs. B.2 and B.3 the right side can be divided by e and e^2 respectively, and consequently only the four variables X_j/e and R_i/e ($i, j = 1, 2$) are involved.

On the other hand, there are three conditions to be fulfilled

1. MAX β_{31}
2. Equation B.1
3. MAX $\beta_{31} = \beta^* = \beta_+$

This leaves one degree of freedom in the problem and therefore a relationship between R_1/e and R_2/e should be obtained.

Because of its complexity, the problem was solved numerically and the result was plotted in Fig. 10 of the main text as a curve in the $u-v$ plane. This is possible since R_i/e and R_2/e can be expressed in terms of u and v . The interpretation of this curve is that the value of the game with pure attacker strategy is equal to the one with mixed attacker strategy.

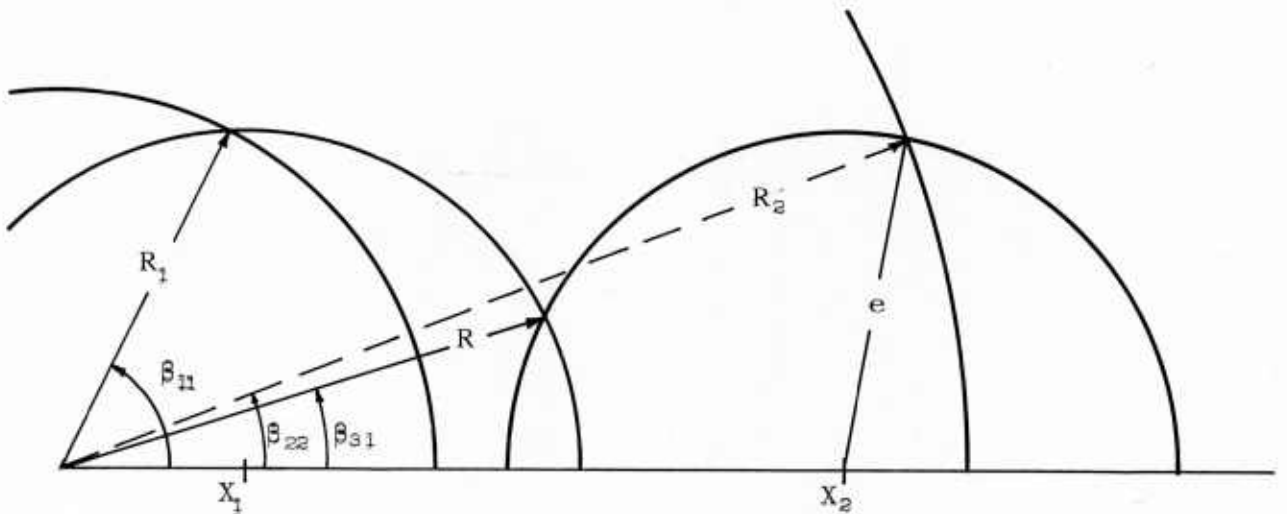


FIG. B.1 GEOMETRY FOR TWO-AIMPOINT STRATEGY

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